

Generalized monogamy inequality of negativity for multi-qubit systems

Wei Chen¹*, Gang Li², Zhu-Jun Zheng²†

¹School of Automation Science and Engineering,

South China University of Technology, Guangzhou 510641, China

²School of Mathematics, South China University of Technology, Guangzhou 510641, China

Abstract

In this paper, we present some generalized monogamy inequalities based on negativity and convex-roof extended negativity (CREN). The monogamy relations are satisfied by the negativity of N -qubit quantum systems $ABC_1 \cdots C_{N-2}$, under the partition AB and $C_1 C_2 \cdots C_{N-2}$. This result can be generalized to the N -qubit pure states under the partition ABC_1 and $C_2 \cdots C_{N-2}$. Furthermore, we give some examples to test the generalized monogamy inequalities.

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One of the most fundamental properties of quantum correlations is that they are not shareable when distributed among many parties. This property distinguishes the quantum correlations from classical correlations. A simple example is a pure maximally entangled state shared between Alice and Bob. This state cannot share any additional correlation (classical or quantum) with other parties. The composite system with a third party, say Carol, can only be a tensor product of the state of Alice and Bob with the state of Carol. This property has been called the monogamy of entanglement and it means that the monogamy relation of entanglement is a way to characterize the different types of entanglement distribution. The monogamy relations give rise to structures of entanglement in multipartite setting and it is important for many tasks in quantum information theory, particularly, in quantum key distribution [1] and quantum correlations [2, 3] like quantum discord[4].

Although it has been shown that quantum correlation measure and entanglement measures cannot satisfy the traditional monogamy relations, it has been shown that it does satisfy the squared concurrence C^2 [5, 6, 7, 8] and the squared entanglement of formation E^2 [7, 8, 9]. Other useful entanglement measures are negativity [10] and convex-roof extended negativity (CREN) [11]. The authors in [12] showed that the monogamy inequality holds in terms of squared negativity for three-qubit states. Kim et al showed that the squared CREN follows the monogamy inequality [13].

In this paper, we study the general monogamy inequalities of CREN in multi-qubit systems. We first recall some basic concepts of entanglement measures. Then we find

*e-mail: auwchen@scut.edu.cn

†e-mail: zhengzj@scut.edu.cn

that the generalized monogamy inequalities always hold based on negativity and CREN in multi-qubit systems. Detailed examples are given to test the generalized monogamy inequalities. We then generalize our results to the N -qubit pure states under the partition ABC_1 and $C_2 \cdots C_{N-2}$.

We first recall the definition of concurrence, negativity and CREN. Given a bipartite pure state $|\psi\rangle_{AB}$ in a $d \otimes d'$ ($d \leq d'$) quantum system, its concurrence, $C(|\psi\rangle_{AB})$ is defined as [14]

$$\mathcal{C}(|\psi\rangle_{AB}) = \sqrt{2[1 - \text{Tr}(\rho_A^2)]} = \sqrt{2[1 - \text{Tr}(\rho_B^2)]}, \quad (1)$$

where ρ_A is reduced density matrix by tracing over the subsystem B , $\rho_A = \text{Tr}_B(|\psi\rangle_{AB}\langle\psi|)$ (and analogously for ρ_B). For any mixed state ρ_{AB} , the concurrence is given by the minimum average concurrence taken over all decompositions of ρ_{AB} , the so-called convex roof [15]

$$\mathcal{C}(\rho_{AB}) = \min_{\{p_i, |\psi_i\rangle\}} \sum_i p_i \mathcal{C}(|\psi_i\rangle), \quad (2)$$

where the convex roof is notoriously hard to evaluate and therefore it is difficult to determine whether or not an arbitrary state is entangled.

Similarly, the COA of any mixed state ρ_{AB} is defined as [16]

$$\mathcal{C}_a(\rho_{AB}) = \max_{\{p_i, |\psi_i\rangle\}} \sum_i p_i \mathcal{C}(|\psi_i\rangle), \quad (3)$$

where the maximum is taken over all possible pure state decompositions $\{p_i, |\psi_i\rangle\}$ of ρ_{AB} .

Another well-known quantification of bipartite entanglement is negativity [10], which is based on the positive partial transposition (PPT) criterion [17, 18]. For a bipartite state ρ_{AB} in a $d \otimes d'$ ($d \leq d'$) quantum system, its negativity is defined as

$$\mathcal{N}(\rho_{AB}) = \frac{\|\rho_{AB}^{T_A}\| - 1}{2}, \quad (4)$$

where $\rho_{AB}^{T_A}$ is the partial transpose with respect to the subsystem A and $\|X\|$ denotes the trace norm of X , i.e. $\|X\| = \text{Tr}\sqrt{XX^\dagger}$. For the purposes of discussion, we use the following definition of negativity:

$$\mathcal{N}(\rho_{AB}) = \|\rho_{AB}^{T_A}\| - 1, \quad (5)$$

Lemma 1. [13] *For a bipartite pure state $|\psi\rangle_{AB}$ in a $d \otimes d'$ ($d \leq d'$) quantum system with the Schmidt decomposition,*

$$|\psi\rangle_{AB} = \sum_{i=0}^{d-1} \sqrt{\lambda_i} |ii\rangle, \quad \lambda_i \geq 0, \quad \sum_{i=0}^{d-1} \lambda_i = 1, \quad (6)$$

(without loss of generality, the Schmidt basis is taken to be the standard basis), we have

$$\mathcal{N}(\rho_{AB}) = 2 \sum_{i < j} \sqrt{\lambda_i \lambda_j}. \quad (7)$$

One modification of negativity to overcome its lack of separability criterion is CREN, which gives a perfect discrimination of PPT bound entangled states and separable states in any bipartite quantum system. For a bipartite mixed state ρ_{AB} , CREN is defined as

$$\tilde{\mathcal{N}}(\rho_{AB}) = \min_{\{p_i, |\psi_i\rangle\}} \sum_i p_i \mathcal{N}(|\psi_i\rangle), \quad (8)$$

where the minimum is taken over all possible pure state decompositions $\{p_i, |\psi_i\rangle\}$ of ρ_{AB} .

Similar to the duality between concurrence and COA, we can also define a dual to CREN, namely CRENOA, by taking the maximum value of average negativity over all possible pure state decomposition, i.e.

$$\tilde{\mathcal{N}}_a(\rho_{AB}) = \max_{\{p_i, |\psi_i\rangle\}} \sum_i p_i \mathcal{N}(|\psi_i\rangle), \quad (9)$$

where the maximum is taken over all possible pure state decompositions $\{p_i, |\psi_i\rangle\}$ of ρ_{AB} .

Theorem 1. *For any $2 \otimes 2 \otimes \cdots \otimes 2 \otimes 2$ pure state $|\psi\rangle_{ABC_1C_2\cdots C_{N-2}}$, we have*

$$\mathcal{N}^2(|\psi\rangle_{AB|C_1C_2\cdots C_{N-2}}) \geq \max\left\{\sum_{i=1}^{N-2} [\tilde{\mathcal{N}}^2(\rho_{AC_i}) - \tilde{\mathcal{N}}_a^2(\rho_{BC_i})], \sum_{i=1}^{N-2} [\tilde{\mathcal{N}}^2(\rho_{BC_i}) - \tilde{\mathcal{N}}_a^2(\rho_{AC_i})]\right\}, \quad (10)$$

where $\rho_{AB} = \text{Tr}_{C_1C_2\cdots C_{N-2}}(|\psi\rangle\langle\psi|)$, $\rho_{AC_i} = \text{Tr}_{BC_1C_2\cdots C_{i-1}C_{i+1}\cdots C_{N-2}}(|\psi\rangle\langle\psi|)$ and $\rho_{BC_i} = \text{Tr}_{AC_1C_2\cdots C_{i-1}C_{i+1}\cdots C_{N-2}}(|\psi\rangle\langle\psi|)$.

Proof. For any $2 \otimes 2 \otimes \cdots \otimes 2 \otimes 2$ pure state $|\psi\rangle_{ABC_1C_2\cdots C_{N-2}}$, we have a Schmidt decomposition $|\psi\rangle_{AB|C_1C_2\cdots C_{N-2}} = \sum_{i=0}^3 \sqrt{\lambda_i} |ii\rangle$. Then from (1), we get

$$\mathcal{C}(|\psi\rangle_{AB|C_1C_2\cdots C_{N-2}}) = \sqrt{2(1 - \text{Tr}\rho_{AB}^2)} \quad (11)$$

where

$$\begin{aligned} \rho_{AB} &= \text{Tr}_{C_1C_2\cdots C_{N-2}}(|\psi\rangle_{AB|C_1C_2\cdots C_{N-2}}\langle\psi|) \\ &= \text{Tr}_{C_1C_2\cdots C_{N-2}}\left(\sum_{i=0}^3 \sqrt{\lambda_i} |ii\rangle \sum_{j=0}^3 \sqrt{\lambda_j} \langle jj|\right) \\ &= \sum_{i=0}^3 \lambda_i |i\rangle\langle i|. \end{aligned}$$

We have $\rho_{AB}^2 = \sum_{i=0}^3 \lambda_i^2 |i\rangle\langle i|$, and we obtain

$$\mathcal{C}(|\psi\rangle_{AB|C_1C_2\cdots C_{N-2}}) = \sqrt{4(\lambda_0\lambda_1 + \lambda_0\lambda_2 + \lambda_0\lambda_3 + \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3)}. \quad (12)$$

From Lemma 1, we have

$$\mathcal{N}(|\psi\rangle_{AB|C_1C_2\cdots C_{N-2}}) = 2(\sqrt{\lambda_0\lambda_1} + \sqrt{\lambda_0\lambda_2} + \sqrt{\lambda_0\lambda_3} + \sqrt{\lambda_1\lambda_2} + \sqrt{\lambda_1\lambda_3} + \sqrt{\lambda_2\lambda_3}). \quad (13)$$

Associated with Theorem 1 in [19], we prove

$$\begin{aligned} \mathcal{N}^2(|\psi\rangle_{AB|C_1C_2\cdots C_{N-2}}) &\geq \mathcal{C}^2(|\psi\rangle_{AB|C_1C_2\cdots C_{N-2}}) \\ &\geq \max\left\{\sum_{i=1}^{N-2} [\mathcal{C}^2(\rho_{AC_i}) - \mathcal{C}_a^2(\rho_{BC_i})], \sum_{i=1}^{N-2} [\mathcal{C}^2(\rho_{BC_i}) - \mathcal{C}_a^2(\rho_{AC_i})]\right\}. \\ &= \max\left\{\sum_{i=1}^{N-2} [\tilde{\mathcal{N}}^2(\rho_{AC_i}) - \tilde{\mathcal{N}}_a^2(\rho_{BC_i})], \sum_{i=1}^{N-2} [\tilde{\mathcal{N}}^2(\rho_{BC_i}) - \tilde{\mathcal{N}}_a^2(\rho_{AC_i})]\right\}, \end{aligned}$$

where the relations $\mathcal{C}(\rho_{AC_i}) = \tilde{\mathcal{N}}(\rho_{AC_i})$, $\mathcal{C}_a(\rho_{BC_i}) = \tilde{\mathcal{N}}_a(\rho_{BC_i})$, $\mathcal{C}(\rho_{BC_i}) = \tilde{\mathcal{N}}(\rho_{BC_i})$, $\mathcal{C}_a(\rho_{AC_i}) = \tilde{\mathcal{N}}_a(\rho_{AC_i})$ in [13] have been used in second equality. Therefore, we have (10).

Theorem 1 shows that the entanglement contained in the pure states $|\psi\rangle_{ABC_1C_2\cdots C_{N-2}}$ is related to the sum of entanglement between bipartitions of the system.

The lower bound in inequalities (18) is easily calculated. As an example, let us consider the four-qubit pure state

$$|\psi\rangle_{ABCD} = \frac{1}{\sqrt{2}}(|0000\rangle + |1001\rangle). \quad (14)$$

Then we have $\rho_{AC} = \frac{1}{2}(|00\rangle\langle 00| + |10\rangle\langle 10|)$, $\rho_{AD} = \frac{1}{2}(|00\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 11|)$, $\rho_{BC} = |00\rangle\langle 00|$, $\rho_{BD} = \frac{1}{2}(|00\rangle\langle 00| + |01\rangle\langle 01|)$, and we obtain $\tilde{\mathcal{N}}(\rho_{AC}) = 0$, $\tilde{\mathcal{N}}(\rho_{AD}) = 1$, $\tilde{\mathcal{N}}_a(\rho_{BC}) = \tilde{\mathcal{N}}_a(\rho_{BD}) = 0$, while we can compute $\mathcal{N}(|\psi\rangle_{AB|CD}) = 1$. Therefore we show that (10) is right for the four-qubit pure state.

Theorem 1 gives a monogamy-type lower bound of $\mathcal{N}(|\psi\rangle_{AB|C_1C_2\cdots C_{N-2}})$. According to the relation between negativity and concurrence, we also have the following theorem:

Theorem 2. *For any $2 \otimes 2 \otimes \cdots \otimes 2 \otimes 2$ pure state $|\psi\rangle_{ABC_1C_2\cdots C_{N-2}}$, we have*

$$\mathcal{N}^2(|\psi\rangle_{AB|C_1C_2\cdots C_{N-2}}) \leq \sqrt{\frac{r(r-1)}{2}} [2\tilde{\mathcal{N}}_a^2(\rho_{AB}) + \sum_{i=1}^{N-2} (\tilde{\mathcal{N}}_a^2(\rho_{AC_i}) + \tilde{\mathcal{N}}_a^2(\rho_{BC_i}))], \quad (15)$$

where $\rho_{AB} = \text{Tr}_{C_1C_2\cdots C_{N-2}}(|\psi\rangle\langle\psi|)$, $\rho_{AC_i} = \text{Tr}_{BC_1C_2\cdots C_{i-1}C_{i+1}\cdots C_{N-2}}(|\psi\rangle\langle\psi|)$, $\rho_{BC_i} = \text{Tr}_{AC_1C_2\cdots C_{i-1}C_{i+1}\cdots C_{N-2}}(|\psi\rangle\langle\psi|)$, and r is the Schmidt rank (the number of nonvanishing Schmidt coefficients λ_j) of the pure state $|\psi\rangle_{AB|C_1C_2\cdots C_{N-2}}$.

Proof. For any $2 \otimes 2 \otimes \cdots \otimes 2 \otimes 2$ pure state $|\psi\rangle_{ABC_1C_2\cdots C_{N-2}}$, then from (32) in [20], we have

$$\mathcal{N}(|\psi\rangle_{AB|C_1C_2\cdots C_{N-2}}) \leq \sqrt{\frac{r(r-1)}{2}} \mathcal{C}(|\psi\rangle_{AB|C_1C_2\cdots C_{N-2}}) \quad (16)$$

From Theorem 2 in [19], we know that

$$\mathcal{C}^2(|\psi\rangle_{AB|C_1C_2\cdots C_{N-2}}) \leq 2\mathcal{C}_a^2(\rho_{AB}) + \sum_{i=1}^{N-2} (\mathcal{C}_a^2(\rho_{AC_i}) + \mathcal{C}_a^2(\rho_{BC_i})), \quad (17)$$

By using the following relations in [13]:

$$\mathcal{C}_a(\rho_{AB}) = \tilde{\mathcal{N}}_a(\rho_{AB}), \mathcal{C}_a(\rho_{AC_i}) = \tilde{\mathcal{N}}_a(\rho_{AC_i}), \mathcal{C}(\rho_{BC_i}) = \tilde{\mathcal{N}}_a(\rho_{BC_i}).$$

We can obtain (15).

For the four-qubit state (14), we have $\tilde{\mathcal{N}}_a(\rho_{AC}) = \tilde{\mathcal{N}}_a(\rho_{AB}) = \tilde{\mathcal{N}}_a(\rho_{BC}) = \tilde{\mathcal{N}}_a(\rho_{BD}) = 0$ and $\tilde{\mathcal{N}}_a(\rho_{AD}) = 1$, while we can compute $\mathcal{N}(|\psi\rangle_{AB|CD}) = 1$. The state also saturates the inequality (15).

Particularly, when $r = 2$, we have the following corollary:

Corollary 1. *For any $2 \otimes 2 \otimes \cdots \otimes 2 \otimes 2$ pure state $|\psi\rangle_{ABC_1C_2\cdots C_{N-2}}$, we have*

$$\mathcal{N}^2(|\psi\rangle_{AB|C_1C_2\cdots C_{N-2}}) \leq 2\tilde{\mathcal{N}}_a^2(\rho_{AB}) + \sum_{i=1}^{N-2} (\tilde{\mathcal{N}}_a^2(\rho_{AC_i}) + \tilde{\mathcal{N}}_a^2(\rho_{BC_i})), \quad (18)$$

where $\rho_{AB} = \text{Tr}_{C_1C_2\cdots C_{N-2}}(|\psi\rangle\langle\psi|)$, $\rho_{AC_i} = \text{Tr}_{BC_1C_2\cdots C_{i-1}C_{i+1}\cdots C_{N-2}}(|\psi\rangle\langle\psi|)$, and $\rho_{BC_i} = \text{Tr}_{AC_1C_2\cdots C_{i-1}C_{i+1}\cdots C_{N-2}}(|\psi\rangle\langle\psi|)$.

From Theorem 1 and Corollary 1, we have

$$\begin{aligned} & |\mathcal{N}^2(|\psi\rangle_{A|BC_1C_2\cdots C_{N-2}}) - \mathcal{N}^2(|\psi\rangle_{B|AC_1C_2\cdots C_{N-2}})| \\ & \leq \mathcal{N}^2(|\psi\rangle_{AB|C_1C_2\cdots C_{N-2}}) \\ & \leq \mathcal{N}^2(|\psi\rangle_{A|BC_1C_2\cdots C_{N-2}}) + \mathcal{N}^2(|\psi\rangle_{B|AC_1C_2\cdots C_{N-2}}). \end{aligned} \quad (19)$$

This implies that if the systems B and $AC_1C_2\cdots C_{N-2}$ are not entangled, the entanglement between the systems AB and $C_1C_2\cdots C_{N-2}$ is equal to the entanglement between the systems A and $BC_1C_2\cdots C_{N-2}$ for pure states $|\psi\rangle_{ABC_1C_2\cdots C_{N-2}}$.

Hence we have

$$\mathcal{N}^2(|\psi\rangle_{A|BC_1C_2\cdots C_{N-2}}) \leq \mathcal{N}^2(|\psi\rangle_{B|AC_1C_2\cdots C_{N-2}}) + \mathcal{N}^2(|\psi\rangle_{AB|C_1C_2\cdots C_{N-2}}),$$

and

$$\mathcal{N}^2(|\psi\rangle_{B|AC_1C_2\cdots C_{N-2}}) \leq \mathcal{N}^2(|\psi\rangle_{A|BC_1C_2\cdots C_{N-2}}) + \mathcal{N}^2(|\psi\rangle_{AB|C_1C_2\cdots C_{N-2}}),$$

and by using Theorem 1, we know that the sum of any two $\mathcal{N}^2(|\psi\rangle_{A|BC_1C_2\cdots C_{N-2}})$, $\mathcal{N}^2(|\psi\rangle_{B|AC_1C_2\cdots C_{N-2}})$, $\mathcal{N}^2(|\psi\rangle_{AB|C_1C_2\cdots C_{N-2}})$ is greater than or equal to the third.

Now we consider further the generalized monogamy relations in terms of arbitrary partitions for the N -qubit generalized W -class states [21]:

$$|W\rangle_{A_1A_2\cdots A_N} = a_1|10\cdots 0\rangle_{A_1A_2\cdots A_N} + a_2|01\cdots 0\rangle_{A_1A_2\cdots A_N} + \cdots + a_N|00\cdots 1\rangle_{A_1A_2\cdots A_N}, \quad (20)$$

where $\sum_{i=1}^N |a_i|^2 = 1$. We have

$$\mathcal{N}^2(|\psi\rangle_{A_1A_2|\cdots A_N}) = 4(a_1^2 + a_2^2)\left(\sum_{i=3}^N a_i^2\right),$$

$$\mathcal{N}^2(|\psi\rangle_{A_1|A_2\cdots A_N}) = 4a_1^2\left(\sum_{i=2}^N a_i^2\right),$$

$$\mathcal{N}^2(|\psi\rangle_{A_2|A_1\cdots A_N}) = 4a_2^2\left(a_1^2 + \sum_{i \neq 3}^N a_i^2\right),$$

and the relation (19) is satisfied. (15) is also correct for the N -qubit generalized W -class states.

We can generalize our results to the negativity $\mathcal{N}(|\psi\rangle_{ABC_1|C_2\cdots C_{N-2}})$ under partition ABC_1 and $C_2C_3\cdots C_{N-2}$ for pure state $|\psi\rangle_{ABC_1C_2\cdots C_{N-2}}$, similar to the theorem 1 and the theorem 2, we can get some generalized results as follow:

Theorem 3. For any $2 \otimes 2 \otimes \cdots \otimes 2 \otimes 2$ pure state $|\psi\rangle_{ABC_1C_2\cdots C_{N-2}}$, we have

$$\begin{aligned} & \mathcal{N}^2(|\psi\rangle_{ABC_1|C_2\cdots C_{N-2}}) \\ & \geq \max\left\{\sum_{i=1}^{N-2} [\tilde{\mathcal{N}}^2(\rho_{AC_i}) - \tilde{\mathcal{N}}_a^2(\rho_{BC_i})], \right. \\ & \quad \left. \sum_{i=1}^{N-2} [\tilde{\mathcal{N}}^2(\rho_{BC_i}) - \tilde{\mathcal{N}}_a^2(\rho_{AC_i})] \right\} - \sum_{j \in J} \tilde{\mathcal{N}}_a^2(\rho_{C_1j}), \end{aligned} \quad (21)$$

where $J = \{A, B, C_2, \cdots, C_{N-2}\}$ and ρ_{C_1j} is the reduced density matrix by tracing over the subsystems except for C_1 and j .

Proof. For any $2 \otimes 2 \otimes \cdots \otimes 2 \otimes 2$ pure state $|\psi\rangle_{ABC_1C_2\cdots C_{N-2}}$, we have a Schmidt decomposition $|\psi\rangle_{ABC_1|C_2\cdots C_{N-2}} = \sum_{i=0}^6 \sqrt{\lambda_i} |ii\rangle$. Then from (1), we get

$$\mathcal{C}(|\psi\rangle_{ABC_1|C_2\cdots C_{N-2}}) = \sqrt{2(1 - \text{Tr} \rho_{ABC_1}^2)} \quad (22)$$

where

$$\rho_{ABC_1} = \text{Tr}_{C_2\cdots C_{N-2}}(|\psi\rangle_{ABC_1|C_2\cdots C_{N-2}}\langle\psi|)$$

$$\begin{aligned}
&= \text{Tr}_{C_2 C_3 \dots C_{N-2}} \left(\sum_{i=0}^6 \sqrt{\lambda_i} |ii\rangle \sum_{j=0}^6 \sqrt{\lambda_j} \langle jj| \right) \\
&= \sum_{i=0}^6 \lambda_i |i\rangle \langle i|,
\end{aligned}$$

and we have $\rho_{ABC_1}^2 = \sum_{i=0}^6 \lambda_i^2 |i\rangle \langle i|$. Hence we obtain

$$\mathcal{C}(|\psi\rangle_{ABC_1|C_2 \dots C_{N-2}}) = \sqrt{4 \left(\sum_{i < j} \lambda_i \lambda_j \right)}. \quad (23)$$

From Lemma 1, we have

$$\mathcal{N}(|\psi\rangle_{ABC_1|C_2 \dots C_{N-2}}) = 2 \left(\sum_{i < j} \sqrt{\lambda_i \lambda_j} \right), \quad (24)$$

and we easily get $\mathcal{N}(|\psi\rangle_{ABC_1|C_2 \dots C_{N-2}}) \geq \mathcal{C}(|\psi\rangle_{ABC_1|C_2 \dots C_{N-2}})$.

Hence, associated with Corollary 1 in [19], we have

$$\begin{aligned}
\mathcal{N}^2(|\psi\rangle_{ABC_1|C_2 \dots C_{N-2}}) &\geq \mathcal{C}^2(|\psi\rangle_{ABC_1|C_2 \dots C_{N-2}}) \\
&\geq \max \left\{ \sum_{i=1}^{N-2} [\mathcal{C}^2(\rho_{AC_i}) - \mathcal{C}_a^2(\rho_{BC_i})], \sum_{i=1}^{N-2} [\mathcal{C}^2(\rho_{BC_i}) - \mathcal{C}_a^2(\rho_{AC_i})] \right\} - \sum_{j \in J} \tilde{\mathcal{N}}_a^2(\rho_{C_{1j}}). \\
&= \max \left\{ \sum_{i=1}^{N-2} [\tilde{\mathcal{N}}^2(\rho_{AC_i}) - \tilde{\mathcal{N}}_a^2(\rho_{BC_i})], \sum_{i=1}^{N-2} [\tilde{\mathcal{N}}^2(\rho_{BC_i}) - \tilde{\mathcal{N}}_a^2(\rho_{AC_i})] \right\} - \sum_{j \in J} \tilde{\mathcal{N}}_a^2(\rho_{C_{1j}}),
\end{aligned}$$

where the relations $\mathcal{C}(\rho_{AC_i}) = \tilde{\mathcal{N}}(\rho_{AC_i})$, $\mathcal{C}_a(\rho_{BC_i}) = \tilde{\mathcal{N}}_a(\rho_{BC_i})$, $\mathcal{C}(\rho_{BC_i}) = \tilde{\mathcal{N}}(\rho_{BC_i})$, $\mathcal{C}_a(\rho_{AC_i}) = \tilde{\mathcal{N}}_a(\rho_{AC_i})$, $\mathcal{C}_a(\rho_{C_{1j}}) = \tilde{\mathcal{N}}_a(\rho_{C_{1j}})$ in [13] have been used in second equality. Therefore, we have (21).

Similarly as the proof of Theorem 2 and associated with Corollary 2 in [19], we have:

Theorem 4. For any $2 \otimes 2 \otimes \dots \otimes 2 \otimes 2$ pure state $|\psi\rangle_{ABC_1 C_2 \dots C_{N-2}}$, we have

$$\begin{aligned}
\mathcal{N}^2(|\psi\rangle_{ABC_1|C_2 \dots C_{N-2}}) &\leq \sqrt{\frac{r(r-1)}{2}} [2\tilde{\mathcal{N}}_a^2(\rho_{AB}) + \sum_{i=1}^{N-2} \tilde{\mathcal{N}}_a^2(\rho_{AC_i}) \\
&\quad + \sum_{i=1}^{N-2} \tilde{\mathcal{N}}_a^2(\rho_{BC_i}) + \sum_{j \in J} \tilde{\mathcal{N}}_a^2(\rho_{C_{1j}})],
\end{aligned} \quad (25)$$

where J and $\rho_{C_{1j}}$ are defined as in Theorem 3, r is the Schmidt rank of the pure state $|\psi\rangle_{ABC_1|C_2 \dots C_{N-2}}$.

In summary, we have discussed the generalized monogamy relations of negativity for N -qubit systems. The generalized monogamy inequalities provide the lower and upper bounds of $\mathcal{N}(|\psi\rangle_{AB|C_1 C_2 \dots C_{N-2}})$ based on bipartite entanglement $\tilde{\mathcal{N}}(\rho_{AC_i})$, $\tilde{\mathcal{N}}(\rho_{BC_i})$, $\tilde{\mathcal{N}}_a(\rho_{AB})$, $\tilde{\mathcal{N}}_a(\rho_{AC_i})$, $\tilde{\mathcal{N}}_a(\rho_{BC_i})$, which are showed by using the CREN and the CRENOA. When the Schmidt rank $r = 2$, Theorem 1 and Theorem 2 gives a lower and upper bounds of $\mathcal{N}^2(|\psi\rangle_{AB|C_1 C_2 \dots C_{N-2}})$ by using the triangle inequality among $\mathcal{N}^2(|\psi\rangle_{AB|C_1 C_2 \dots C_{N-2}})$, $\mathcal{N}^2(|\psi\rangle_{A|BC_1 C_2 \dots C_{N-2}})$ and $\mathcal{N}^2(|\psi\rangle_{B|AC_1 C_2 \dots C_{N-2}})$. Some examples are given to show the generalized monogamy relations and we generalized these results to $\mathcal{N}(|\psi\rangle_{ABC_1|C_2 \dots C_{N-2}})$, and we obtained Theorem 3 and Theorem 4.

Entanglement monogamy is a fundamental property of multipartite entangled states. The generalized monogamy relations maybe test some higher-dimensional quantum systems. We believe that these generalized monogamy inequalities can be useful in quantum information theory. When we complete our paper, we find that some results in this paper were discussed in [22]. But the proofs in [22] are valid only for Schmidt rank two. If Schmidt rank is not two, the theorem 2 in [22] is not correct.

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